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New explicit exact solutions for the generalized coupled Hirota–Satsuma KdV system

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Abstract

In this paper, we study the generalized coupled Hirota–Satsuma KdV system by using the two new improved projective Riccati equations method. As a result, many explicit exact solutions, which contain new solitary wave solutions, periodic wave solutions and combined formal solitary wave solutions and combined formal periodic wave solutions are obtained.

Keywords: Generalized coupled Hirota–Satsuma KdV system; Riccati equations; Solitary wave solution; Periodic wave solution

1. Introduction

In recent years, due to the wide applications of soliton theory in mathematics, physics, chemistry, biology, communications, astrophysics and geophysics, etc., the search for explicit exact solutions, in particular, solitary wave solutions of nonlinear evolution equations (NEEs) has played an important role in the soliton theory. Various effective methods have been developed, such as inverse scattering transformation, Hirota bilinear method, Backlund transformation, Darboux transformation, tanh-function method, extended tanh-function method, sine–cosine method, homogeneous balance method, Jacobian elliptical function expansion method and its generalization, Li group analysis, similarity reduced method, F-expansion method, transformation methods in terms of the Weierstrass elliptical function solutions, and so on.

In 1992, Conte and Musette [1] presented a projective Riccati equation method to seek more new solitary wave solutions to NEEs that can be expressed as polynomial in two elementary functions which satisfy a projective Riccati equation [2]. The method had been applied to find many solitary wave solutions of many equations. In this paper, we will construct two new Riccati equations to generalize the Riccati method. In illustration, we will obtain several new families of exact soliton solutions for the generalized coupled Hirota–Satsuma KdV system.

This paper is arranged as follows. In Section 2, we briefly describe the new extended projected Riccati equation method. In Section 3, several families of solutions to the generalized coupled Hirota–Satsuma KdV system are

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obtained, including new solitary wave solutions and new periodic wave solutions. In Section 4, some conclusions are given.

2. Summary of the new projected Riccati equation method

For a given partial differential equation, say, in two variables x and t

$$P(u, u_t, u_x, u_{xx}, \dots) = 0. \tag{2.1}$$

We seek the following formal solutions of the given system by a new intermediate transformation:

$$u(x,t) = \sum_{i=0}^{n} A_i f^i(\xi) + \sum_{j=1}^{n} B_j f^{j-1}(\xi) g^j(\xi) \tag{2.2}$$

where $A_0, A_i, B_j, (i, j = 1, 2, \dots n)$ are constants to be determined later, $\xi = \xi(x,t)$ is arbitrary functions with the variables x and t. The parameter n can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq. (2.1). $f(\xi), g(\xi)$ satisfy the following projective Riccati equations:

$$\text{(I)}\ f'(\xi) = -qf(\xi)g(\xi), \qquad g'(\xi) = q[1 - g^2(\xi) - rf(\xi)], \qquad g^2(\xi) = 1 - 2rf(\xi) + (r^2 + \varepsilon)f^2(\xi) \tag{2.3}$$

where "′" denotes $\frac{\mathrm{d}}{\mathrm{d}\xi}$, $\varepsilon = \pm 1$, r, q are arbitrary constants. It is easy to see that Eq. (2.3) admit the following solutions:

$$f_1(\xi) = \frac{a}{b\cosh(q\xi) + c\sinh(q\xi) + ar}, \qquad g_1(\xi) = \frac{b\sinh(q\xi) + c\cosh(q\xi)}{b\cosh(q\xi) + c\sinh(q\xi) + ar} \tag{2.4}$$

when $\varepsilon = 1$: a, b, c satisfies $c^2 = a^2 + b^2$. When $\varepsilon = -1$: a, b, c satisfies $b^2 = a^2 + c^2$.

$$\text{(II)}\ f'(\xi) = qf(\xi)g(\xi), \qquad g'(\xi) = q[1 + g^2(\xi) - rf(\xi)], \qquad g^2(\xi) = -1 + 2rf(\xi) + (1 - r^2)f^2(\xi). \tag{2.5}$$

Eq. (2.5) have the following solutions:

$$f_2(\xi) = \frac{a}{b\cos(q\xi) + c\sin(q\xi) + ar}, \qquad g_2(\xi) = \frac{b\sin(q\xi) - c\cos(q\xi)}{b\cos(q\xi) + c\sin(q\xi) + ar} \tag{2.6}$$

where a, b, c satisfies $a^2 = b^2 + c^2$. Substituting (2.2) with (2.2) and (2.3) with (2.5) into Eq. (2.1) separately yields a set of differential equations for $f^i(\xi)g^j(\xi)$ $(i, j = 1, 2, \dots)$. Setting the coefficients of $f^i(\xi)g^j(\xi)$ to zero yields a set of over-determined differential equations (ODEs) in A_0, A_i, B_j, $(i, j = 1, 2, \dots n)$ and $\xi(x,t)$, solving the ODEs by Mathematica and Wu elimination, we can obtain many exact solutions of Eq. (2.1) according to (2.4) and (2.6).

Obviously, if we choose the special value of a, b, c, q, r in (2.4) and (2.6), then we can get the result of [3–11]. For example, when we choose $b = a = q = 1, c = 0$; $c = a = q = 1, b = 0$ and $q = 1, b = 5, a = 4, c = 3$, then we have $f_{11}(\xi) = \frac{1}{\cosh(\xi)+r}$, $g_{11}(\xi) = \frac{\sinh\xi}{\cosh(\xi)+r}$, $f_{12}(\xi) = \frac{1}{\sinh(\xi)+r}$, $g_{12}(\xi) = \frac{\cosh\xi}{\sinh(\xi)+r}$, $f_{13}(\xi) = \frac{4}{5\cosh(\xi)+3\sinh(\xi)+4r}$, $g_{13}(\xi) = \frac{5\sinh\xi+3\cosh\xi}{5\cosh(\xi)+3\sinh\xi+4r}$, it is the case of [3–7]. If we change the form of (2.4) and (2.6) into $\operatorname{sech}(\xi)$, $\operatorname{csch}(\xi)$, $\tanh(\xi)$, $\coth(\xi)$, $\sec(\xi)$, $\csc(\xi)$, $\tan(\xi)$, $\cot(\xi)$ type, and choose $b = a = 1$, $q = \sqrt{r_1}, r = \mu, c = 0$; $q = \sqrt{r_1}, r = \mu, b = 0, c = a = 1$ there are only a constant times difference between $f_1(\xi), f_2(\xi), g_1(\xi), g_2(\xi)$ and the σ_i, τ_i $(i = 1, 2, 3, 4)$ in [8,9]. It can be unified completely from the coefficient of $f^i(\xi)g^j(\xi)$ in the assumption form of $u(\xi)$, if we choose $a = 1, q = \sqrt{-pq_1}, b = q_1 l, c = q_1 k$; $a = 1, q = \sqrt{pq_1}, b = q_1 l, c = q_1 k$, so does $f_1(\xi), f_2(\xi), g_1(\xi), g_2(\xi)$ and $f_i(\xi)g_i(\xi)$ $(i = 1, 2)$ in [10] thus (2.4) and (2.6) contain the case of [3–10] completely just the result turns more commonly, the form turns more simply; here many solutions are new.

If we choose $\varepsilon = -1, r = c = 0, b = a, q = 1$ in (2.4), we can obtain the bell-type solitary wave solutions and kink solitary wave solution: $f_{1'}(\xi) = \operatorname{sech}\xi$, $g_{1'}(\xi) = \tanh\xi$, and if we choose $\varepsilon = 1, r = b = 0, c = a, q = 1$, we can obtain singular wave solutions: $f_{1''}(\xi) = \operatorname{csch}\xi$, $g_{1''}(\xi) = \coth\xi$; in common, if choose the special r, c, b, a, q in (2.6), we have $\sec\xi, \csc\xi, \tan\xi, \cot\xi$ type trigonometric function periodic solutions.

Remark 1. Riccati equations (2.3) and (2.5) are new; the solutions (2.4) and (2.6) are new too; they contain the results in [3–11] completely, and the form turns much more simply. Their using value turns more strongly.

In the following, we will use this method to solve the generalized coupled Hirota–Satsuma KdV system:

3. Exact solutions to the generalized coupled Hirota–Satsuma KdV system

In 1981, Hirota–Satsuma first proposed the well-known coupled Hirota–Satsuma KdV system [12], which describes interactions of two long waves with different dispersion relations.

$$\begin{cases} u_t = \frac{1}{4}u_{xxx} + 3uu_x - 6vv_x \\ v_t = -\frac{1}{2}v_{xxx} - 3uv_x \end{cases} \tag{3.1}$$

and further showed that the system (3.1) is a special case of the generalized coupled Hirota–Satsuma KdV system [13].

$$\begin{cases} u_t = \frac{1}{4}u_{xxx} + 3uu_x + 3(-v^2 + w)_x \\ v_t = -\frac{1}{2}v_{xxx} - 3uv_x \\ w_t = -\frac{1}{2}w_{xxx} - 3uw_x \end{cases} \tag{3.2}$$

which can be obtained from the four-reduction of KP hierarchy by the dependent variable transform $u = (\ln f)_{xx}$, $v = \frac{1}{2}\frac{f_y}{f}$, $w = \frac{1}{2}\frac{f_{yy}}{f}$ with y being an auxiliary variable. Chen et al. [14] found many solutions for system (3.2) by a method based on a Riccati equation and homogeneous balance method,but the results are wrong because of the (13a) in [14], Fan [15,16] and Yan [17] studied the explicit solutions of system (3.2) by using the Jacobi elliptical function. Hu and Liu [18] investigated the Darboux transformation for the system (3.2), Weiss [19] studied the Painleve property for the system. Some other research for the system could be referred to in [20–26].

Here we study the system (3.2) by the new improved Riccati equations method discussed above.

Let

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi), \qquad w(x,t) = w(\xi), \qquad \xi = 2kx + \lambda t + \xi_0 \tag{3.3}$$

k, λ, ξ_0 are arbitrary constants to be determined later.

Substituting (3.3) into (3.2) yields the following ordinary differential system:

$$\begin{cases} \lambda u' = 2k^3u''' + 6kuu' - 12kvv' + 6kw' \\ \lambda v' = -4k^3v''' - 6kuv' \\ \lambda w' = -4k^3w''' - 6kuw' \end{cases} \tag{3.4}$$

by the homogeneous balance principle we have thus assumed that (3.4) has the following solutions:

$$\begin{aligned} &u = a_0 + a_1 f + a_2 f^2 + a_3 fg + a_4 g, \qquad v = b_0 + b_1 f + b_2 f^2 + b_3 fg + b_4 g \\ &w = c_0 + c_1 f + c_2 f^2 + c_3 fg + c_4 g, \qquad \xi = 2kx + \lambda t + \xi_0 \end{aligned} \tag{3.5}$$

where $u = u(x,t) = u(\xi), v = v(x,t) = v(\xi), w = w(x,t) = w(\xi), f = f(\xi), g = g(\xi)$ and f, g satisfy (2.3) or (2.5). Substituting (2.3) with (3.5) into (3.4) and setting the coefficients of $f^i g^j$ $(i, j = 1, 2, \ldots)$ to zero yields a set of over-determined ordinary differential equations (ODEs) with respect to the unknowns a_i, b_i, c_i $(i = 0, \ldots, 4), k, \lambda, \varepsilon, r$.

$$\begin{aligned} &6k[2a_2a_4 + (a_0a_4 + c_4 - 2b_0b_4)(r^2 + \varepsilon) - 4(b_1b_3 + b_2b_4)] + 3a_3r\lambda - a_4\lambda(r^2 + \varepsilon) \\ &18kr(2b_0b_3 + 2b_1b_4 - a_0a_3 + c_3) - 6k^3q^2r(5a_3 + a_4) + 6a_1k(2a_3 - 3a_4r) = 0, \\ &15kr(2b_1b_3 - a_1a_3) + 6kr^2(a_0a_3 + a_1a_4 - 2b_0b_3 - 2b_1b_4 + c_3) - a_3\lambda\varepsilon - a_3r^2\lambda + 2k^3q^2r(25a_3 - 6a_4r^2) \\ &\quad + 3a_2k(3a_3 - 5a_4r) - 6b_2k(3b_3 - 5b_4r) + 6a_1a_4k\lambda \end{aligned}$$

$$
\begin{aligned}
&\quad + 6a_0a_3k\varepsilon - 12k\varepsilon(b_0b_3 - b_1b_4) + 6c_3k\varepsilon + 4k^3q^2\varepsilon(5a_3 - 3a_4) = 0,\\
&6k(a_0b_1 + a_4b_3 - a_4b_4r) + 4b_1k^3q^2 + b_1\lambda = 0,\\
&6k(a_0a_1 + a_3a_4 - 2b_0b_1 - 2b_3b_4 + c_1) + 2a_1k^3q^2 + 6kr(2b_4^2 - a_4^2) - a_1\lambda = 0,\\
&-24b_3b_4kr + 6a_1k^3q^2r - 3a_4^2kr^2 + 6b_4^2kr^2 + a_2\lambda - 3a^2k\varepsilon + 6b_4^2k\varepsilon = 0,\\
&3a_1a_2 - 6b_1b_2 - 3a_3^2r + 6b_3^2r + (3a_3a_4 - 6b_3b_4 + 2a_1k^2q^2)(r^2 + \varepsilon) = 0,\\
&a_2^2 - 2b_2^2 + (a_3^2 - 2b_3^2 + 4a_3k^2q^2)(r^2 + \varepsilon) - 10a_3k^2q^2r = 0,\\
&6k(b_4ra_1 - b_4a_0r^2 - a_3b_1 + a_1b_3) + 4k^3q^2(15b_3r - 7b_4r^2 - 4b_4\varepsilon) + r\lambda(3b_3 - b_4r) - b_4\varepsilon(6a_0k - \lambda) = 0,\\
&-6a_4b_1k - (b_3 - b_4r)(6a_0k + 4k^3q^2 + \lambda) = 0,\\
&-12a_4k(b_2 - b_1r) + 18a_0b_3kr - 12a_4b_2kr - 9a_1b_3kr + 3kr^2(a_4b_1 + 2a_0b_3 + a_1b_4)\\
&\quad + 2k^3q^2r^2(50b_3 - 12b_4r) + 6a_3b_2k - 6a_3b_1kr + 3a_2k(b_3 - b_4r) + b_3\lambda(r^2 + \varepsilon)\\
&\quad + 3k\varepsilon(a_4b_1 + 2a_0b_3 + a_1b_4) + 8k^3q^2\varepsilon(5b_3 - 3b_4) = 0,\\
&-3a_2b_3r + 2a_1b_3r^2 + a_2b_4r^2 - 40b_3k^2q^2r^3 + 4b_4k^2q^2r^4 + 2a_1b_3\varepsilon + a_2b_4\varepsilon - 4a_3b_2r - 40b_3k^3q^2r\varepsilon\\
&\quad + 8b_4k^2q^2r^2\varepsilon + 4b_4k^2q^2 + (r^2 + \varepsilon)(2a_4b_2 + a_3b_1) = 0,\\
&a_2b_1 + 2a_1b_2 - 3a_3b_3r - 20b_3k^2q^2r + (2a_4b_3 + a_3b_4 + 4b_1k^2q^2)(r^2 + \varepsilon) = 0,\\
&a_2b_2 + b_3(a_3 + 8k^2q^2)(r^2 + \varepsilon) = 0,\\
&-6a_3c_1k - 12a_4c_2k - 6a_1c_3k + 12a_4c_1kr + 18a_0c_3kr + 6a_1c_4kr + 60c_3k^3q^2r\\
&\quad - 6a_0c_4kr^2 - 28c_4k^3q^2r^2 + 3c_3r\lambda - c_4r^2\lambda - 6a_0c_4k\varepsilon - 16c_4k^3q^2\varepsilon - c_4\lambda\varepsilon = 0,\\
&3kr^2(2a_0c_3 + a_4c_1 + a_1c_4) - 3kr(4a_4c_2 + 3a_1c_3) + 4k^3q^2r^2(25c_3 - 6c_4r) + 6ka_3(c_2 - c_1r)\\
&\quad + 3a_2k(c_3 - c_4r) + c_3\lambda(r^2 + \varepsilon) + 3k\varepsilon(a_4c_1 + 2a_0c_3 + a_1c_4) + 8k^3q^2\varepsilon(5c_3 - 3c_4r) = 0,\\
&a_2c_2 + c_3(r^2 + \varepsilon)(a_3 + 8k^2q^2) = 0,\\
&-3a_2c_3r + 2a_1c_3r^2 + a_2c_4r^2 - 40c_3k^2q^2r^3 + 4c_4k^2q^2r^4 + 2a_1c_3\varepsilon + a_2c_4\varepsilon\\
&\quad + 8k^2q^2r\varepsilon(c_4r - 5c_3k) + 4c_4k^2q^2 - 4a_3c_2r + (r^2 + \varepsilon)(2a_4c_2 + a_3c_1) = 0,\\
&(r^2 + \varepsilon)[a_3c_2 + c_3a_2 + 8c_3k^2q^2(r^2 + \varepsilon)] = 0,\\
&3k(a_1c_1 + a_3c_3 + a_4c_4\varepsilon) + c_2\lambda + 3k(2a_0c_2 - 3a_4c_3r) + 4k^3q^2(4c_3 - 3c_1r) - 3c_4kr(a_3 - a_4r) = 0,\\
&6a_0c_1k + 6a_4c_3k - 6a_4c_4kr + 4c_1k^3q^2 + c_1\lambda = 0,\\
&a_2c_1 + 2a_1c_2 - 3a_3c_3r - 20c_3k^2q^2r + (r^2 + \varepsilon)(2a_4c_3 + 4c_1k^2q^2 + a_3c_4) = 0\\
&6k(a_1a_4 + c_3 - c_4r) - 12k(b_0b_3 + b_1b_4 - b_0b_4r) + (a_3 - a_4r)(2k^3q^2 + 6a_0k - \lambda) = 0,\\
&-3a_1^2k - 6a_0a_2k - 3a_3^2k + 6k(b_1^2 + 2b_0b_2 + b_3^2 - c_2) - 8a_3k^3q^2 + 12a_3a_4kr\\
&\quad \times (3a_1a_3 - 7a_3a_2r + 3a_2a_4)(r^2 + \varepsilon) + (r^2 + \varepsilon)[-6b_1b_3 - 20a_3rk^2q^2 - 14b_2b_4r\\
&\quad + 2k^2q^2a_4(r^2 + \varepsilon)] + 14b_2b_3r = 0,\\
&(r^2 + \varepsilon)[a_2a_3 - 2b_2b_3 + 2a_3k^2q^2(r^2 + \varepsilon)] = 0,\\
&(r^2 + \varepsilon)[a_3b_2 + b_3a_2 + 8b_3k^2q^2(r^2 + \varepsilon)] = 0,\\
&3k(a_1b_1 + 2a_0b_2 + a_3b_3) - 3kr(3a_4b_3 + a_3b_4) + 4k^3q^2(4b_3 - 3b_1r) + b_2\lambda + 3a_4b_4k(r^2 + \varepsilon) = 0,\\
&-6a_4c_1k - (c_3 - c_4r)(6a_0k + 4k^3q^2 + \lambda) = 0.
\end{aligned}
$$

Solving the ODEs we have

Case 1

$$
\begin{aligned}
&a_2 = a_3 = a_4 = 0, \qquad b_2 = b_3 = b_4 = 0, \qquad c_2 = c_3 = c_4 = 0, \qquad a_1 = 4q^2r, \qquad b_1 = \pm 2q^2r,\\
&c_1 = -4q^2(2a_0 \mp b_0 + q^2)r, \qquad \lambda = -6a_0 - 4q^2, \qquad k = 1, \qquad r^2 = -\varepsilon, \qquad \varepsilon = \pm 1.
\end{aligned}
$$

Case 2

$$
a_1 = a_4 = 0, \qquad b_2 = b_3 = 0, \qquad c_2 = c_3 = 0, \qquad a_2 = a_3 = -2k^2q^2,
$$

$$b_1 = b_4 = \pm\sqrt{2a_0k^2q^2 - k^4q^4}, \qquad c_1 = c_4 = 2b_0b_4, \qquad \lambda = -6a_0k - 4k^3q^2, \qquad r = 0, \qquad \varepsilon = 1.$$

Case 3

$$a_2 = a_3 = a_4 = 0, \qquad b_1 = b_2 = b_3 = 0, \qquad c_1 = c_2 = c_3 = 0, \qquad a_1 = 2k^2q^2\mathrm{i}, \qquad c_4 = 2b_0b_4,$$
$$b_4 = \pm\sqrt{-2a_0k^2q^2 - k^4q^4}, \qquad \lambda = -6a_0k - 4k^3q^2, \qquad r = \pm\mathrm{i} = \pm\sqrt{-1}, \qquad \varepsilon = 1.$$

Case 4

$$a_2 = a_3 = a_4 = 0, \qquad b_1 = b_2 = b_3 = 0, \qquad c_1 = c_2 = c_3 = 0, \qquad a_1 = 2k^2q^2r, \qquad c_4 = 2b_0b_4,$$
$$b_4 = \pm\sqrt{-2a_0k^2q^2 - k^4q^4}, \qquad \lambda = -6a_0k - 4k^3q^2, \qquad r = \pm 1, \qquad \varepsilon = -1$$

a_0, b_0, c_0, k and q are arbitrary constants in all above cases. We could determine the following solitary wave solutions of system (3.2):

$$u_1(x,t) = u_1(\xi_1) = a_0 \pm \frac{4q^2a\sqrt{-\varepsilon}}{b\cosh(q\xi_1) + c\sinh(q\xi_1) \pm a\sqrt{-\varepsilon}}$$
$$v_1(x,t) = v_1(\xi_1) = b_0 \pm \frac{2q^2a\sqrt{-\varepsilon}}{b\cosh(q\xi_1) + c\sinh(q\xi_1) \pm a\sqrt{-\varepsilon}}$$
$$w_1(x,t) = w_1(\xi_1) = c_0 - \frac{4q^2(2a_0 \mp b_0 + q^2)a\sqrt{-\varepsilon}}{b\cosh(q\xi_1) + c\sinh(q\xi_1) \pm a\sqrt{-\varepsilon}},$$
$$\varepsilon = \pm 1,\ \xi_1 = 2kx - (6a_0 + 4q^2)t + \xi_0$$
$$u_2(x,t) = u_2(\xi_2) = a_0 - \frac{2k^2q^2a^2}{[b\cosh(q\xi_2) + c\sinh(q\xi_2)]^2} - \frac{2k^2q^2a[b\sinh(q\xi_2) + c\cosh(q\xi_2)]}{[b\cosh(q\xi_2) + c\sinh(q\xi_2)]^2}$$
$$v_2(x,t) = v_2(\xi_2) = b_0 \pm \frac{a\sqrt{2a_0k^2q^2 - k^4q^4}}{b\cosh(q\xi_2) + c\sinh(q\xi_2)} \pm \frac{\sqrt{2a_0k^2q^2 - k^4q^4}[b\sinh(q\xi_2) + c\cosh(q\xi_2)]}{b\cosh(q\xi_2) + c\sinh(q\xi_2)}$$
$$w_2(x,t) = w_2(\xi_2) = c_0 \pm \frac{2b_0a\sqrt{2a_0k^2q^2 - k^4q^4}}{b\cosh(q\xi_2) + c\sinh(q\xi_2)} \pm \frac{2b_0\sqrt{2a_0k^2q^2 - k^4q^4}[b\sinh(q\xi_2) + c\cosh(q\xi_2)]}{b\cosh(q\xi_2) + c\sinh(q\xi_2)}$$
$$\varepsilon = 1,\ \xi_2 = 2kx - (6a_0k + 4k^3q^2)t + \xi_0$$
$$u_3(x,t) = u_3(\xi_3) = a_0 \pm \frac{2k^2q^2a\mathrm{i}}{b\cosh(q\xi_3) + c\sinh(q\xi_3) \pm a\mathrm{i}}$$
$$v_3(x,t) = v_3(\xi_3) = b_0 \pm \frac{\sqrt{-2a_0k^2q^2 - k^4q^4}[b\sinh(q\xi_3) + c\cosh(q\xi_3)]}{b\cosh(q\xi_3) + c\sinh(q\xi_3) \pm a\mathrm{i}}$$
$$w_3(x,t) = w_3(\xi_3) = c_0 \pm \frac{2b_0\sqrt{-2a_0k^2q^2 - k^4q^4}[b\sinh(q\xi_3) + c\cosh(q\xi_3)]}{b\cosh(q\xi_3) + c\sinh(q\xi_3) \pm a\mathrm{i}}$$
$$\varepsilon = 1,\ \xi_3 = 2kx - (6a_0k + 4k^3q^2)t + \xi_0$$
$$u_4(x,t) = u_4(\xi_4) = a_0 \pm \frac{2k^2q^2a}{b\cosh(q\xi_4) + c\sinh(q\xi_4) \pm a}$$
$$v_4(x,t) = v_4(\xi_4) = b_0 \pm \frac{\sqrt{-2a_0k^2q^2 - k^4q^4}[b\sinh(q\xi_4) + c\cosh(q\xi_4)]}{b\cosh(q\xi_4) + c\sinh(q\xi_4) \pm a}$$
$$w_4(x,t) = w_4(\xi_4) = c_0 \pm \frac{2b_0\sqrt{-2a_0k^2q^2 - k^4q^4}[b\sinh(q\xi_4) + c\cosh(q\xi_4)]}{b\cosh(q\xi_4) + c\sinh(q\xi_4) \pm a}$$
$$\varepsilon = -1,\ \xi_4 = 2kx - (6a_0k - 4k^3q^2)t + \xi_0$$

when $\varepsilon = 1$, a, b, c satisfies $c^2 = a^2 + b^2$ when $\varepsilon = -1$, a, b, c satisfies $b^2 = a^2 + c^2$.

We can gives the numerical simulation of u_1, u_2, and w_4 (see Figs. 1–4).

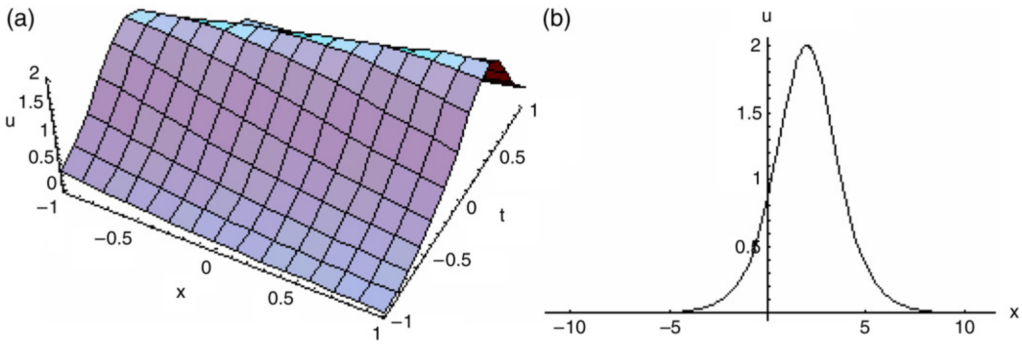

Fig. 1. (a) Bell-shaped solitary wave solutions of u_1 when $c = 0$, $a = 1$, $b = -1$, $a_0 = \xi_0 = 0$, $k = 0.5$, $q = 1$. (b) Plane graph when $t = 0$.

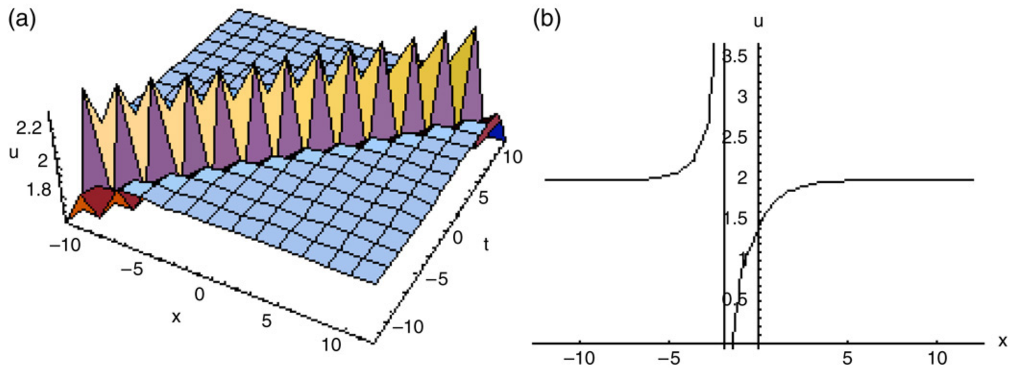

Fig. 2. (a) Singular wave solutions of u_1 when $c = 0$, $a = 1$, $b = -1$, $a_0 = \xi_0 = 0$, $k = 0.5$, $q = 1$. (b) Plane graph when $t = 0$.

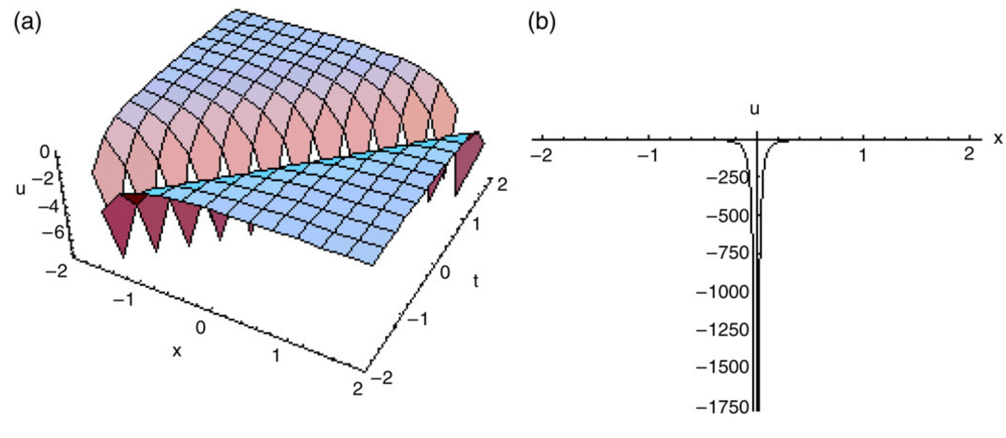

Fig. 3. (a) Singular wave solutions of u_2 when $b = 4$, $a = -3$, $c = 5$, $a_0 = 2$, $\xi_0 = 0$, $k = 0.5$, $q = 1$. (b) Plane graph when $t = 0$.

Similarly, substituting (2.5) along with (3.5) into (3.4), and setting the coefficients of $f^i g^j$ $(i, j = 1, 2, \ldots)$ to zero yields a set of over-determined ordinary differential equations (ODEs) with respect to the unknowns a_i, b_i, c_i $(i = 0, \ldots, 4)$, k, λ, r. Solving this ODEs we have

Case 5

$$a_2 = a_3 = a_4 = 0, \qquad b_2 = b_3 = b_4 = 0, \qquad c_2 = c_3 = c_4 = 0, \qquad a_1 = \mp 4k^2q^2, \qquad b_1 = \pm 2k^2q^2,$$
$$c_1 = \pm 4k^2q^2b_0 \pm 8k^2q^2a_0 \mp 4k^4q^2, \qquad \lambda = 4k^3q^2 - 6a_0k, \qquad r = \pm 1.$$

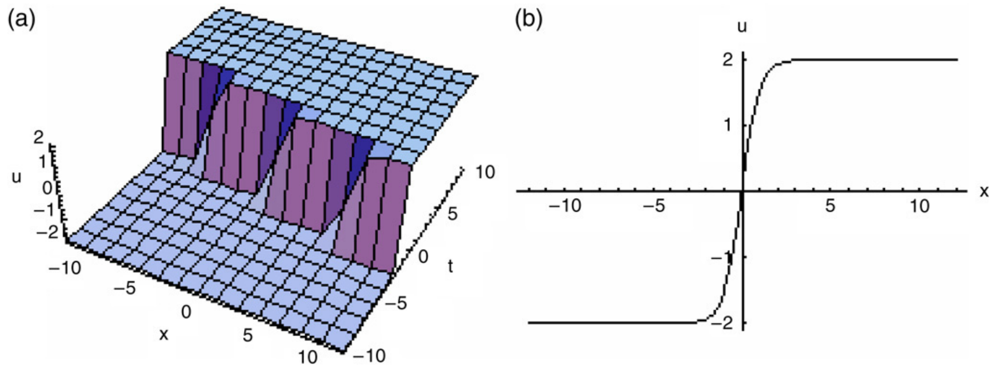

Fig. 4. (a) Solitary wave solutions of w_4 when $c = 0$, $a = b = b_0 = 1$, $a_0 = -1$, $\xi_0 = 0$, $k = 1$, $q = 1$. (b) Plane graph when $t = 0$.

Case 6

$$a_1 = a_4 = 0, \quad b_2 = b_3 = 0, \quad c_2 = c_3 = 0, \quad a_2 = a_3 = -2k^2q^2, b_1 = b_4 = \pm\sqrt{-2a_0k^2q^2 + k^4q^4},$$
$$c_1 = c_4 = 2b_0b_4, \quad \lambda = -6a_0k + 4k^3q^2, \quad r = 0.$$

Case 7

$$a_2 = a_3 = a_4 = 0, \quad b_1 = b_2 = b_3 = 0, \quad c_1 = c_2 = c_3 = 0, \quad a_1 = -2k^2q^2r, \quad c_4 = 2b_0b_4,$$
$$b_4 = \pm\sqrt{-2a_0k^2q^2 + k^4q^4}, \quad \lambda = -6a_0k + 4k^3q^2, \quad r = \pm 1.$$

a_0, b_0, c_0, k and q are arbitrary constants in all the above cases. We could obtain the following periodic wave solutions of the system (3.2):

$$u_5(x,t) = u_5(\xi_5) = a_0 \mp \frac{4k^2q^2a}{b\cos(q\xi_5) + c\sin(q\xi_5) \pm a}$$
$$v_5(x,t) = v_5(\xi_5) = b_0 \pm \frac{2k^2q^2a}{b\cos(q\xi_5) + c\sin(q\xi_5) \pm a}$$
$$w_5(x,t) = w_5(\xi_5) = c_0 \pm \frac{4k^2q^2(2a_0 + b_0 - k^2)a}{b\cos(q\xi_5) + c\sin(q\xi_5) \pm a}$$
$$\xi_5 = 2kx - (6a_0k - 4k^3q^2)t + \xi_0$$
$$u_6(x,t) = u_6(\xi_6) = a_0 - \frac{2k^2q^2a^2}{[b\cos(q\xi_6) + c\sin(q\xi_6)]^2} - \frac{2k^2q^2a[b\sin(q\xi_6) - c\cos(q\xi_6)]}{[b\cos(q\xi_6) + c\sin(q\xi_6)]^2}$$
$$v_6(x,t) = v_6(\xi_6) = b_0 \pm \frac{a\sqrt{-2a_0k^2q^2 + k^4q^4}}{b\cos(q\xi_6) + c\sin(q\xi_6)} \pm \frac{\sqrt{-2a_0k^2q^2 + k^4q^4}[b\sin(q\xi_6) - c\cos(q\xi_6)]}{b\cos(q\xi_6) + c\sin(q\xi_6)}$$
$$w_6(x,t) = w_6(\xi_6) = c_0 \pm \frac{2b_0a\sqrt{-2a_0k^2q^2 + k^4q^4}}{b\cos(q\xi_6) + c\sin(q\xi_6)} \pm \frac{2b_0\sqrt{-2a_0k^2q^2 + k^4q^4}[b\sin(q\xi_6) - c\cos(q\xi_6)]}{b\cos(q\xi_6) + c\sin(q\xi_6)}$$
$$\xi_6 = 2kx - (6a_0k - 4k^3q^2)t + \xi_0$$
$$u_7(x,t) = u_7(\xi_7) = a_0 \pm \frac{2k^2q^2ra}{b\cos(q\xi_7) + c\sin(q\xi_7) \pm a}$$
$$v_7(x,t) = v_5(\xi_7) = b_0 \pm \frac{\sqrt{-2a_0k^2q^2 + k^4q^4}[b\sin(q\xi_7) - c\cos(q\xi_7)]}{b\cos(q\xi_7) + c\sin(q\xi_7) \pm a}$$
$$w_7(x,t) = w_5(\xi_7) = c_0 \pm \frac{2b_0\sqrt{-2a_0k^2q^2 + k^4q^4}[b\sin(q\xi_7) - c\cos(q\xi_7)]}{b\cos(q\xi_7) + c\sin(q\xi_7) \pm a}$$
$$\xi_7 = 2kx - (6a_0k - 4k^3q^2)t + \xi_0$$

the arbitrary constants a, b, c satisfies $a^2 = b^2 + c^2$ in all above cases.

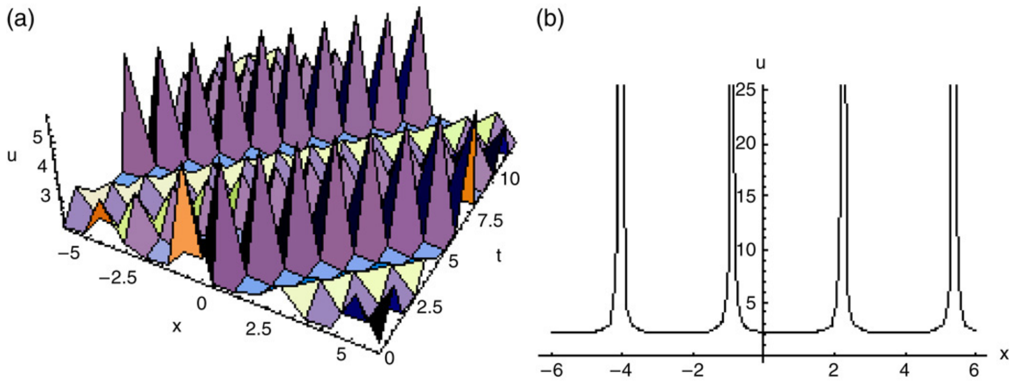

Fig. 5. (a) Periodic wave solutions of u_6 when $a = 5, b = 3, c = 4, a_0 = -\frac{1}{6}, \xi_0 = 0, k = 0.5, q = 1$. (b) Plane graph when $t = 0$.

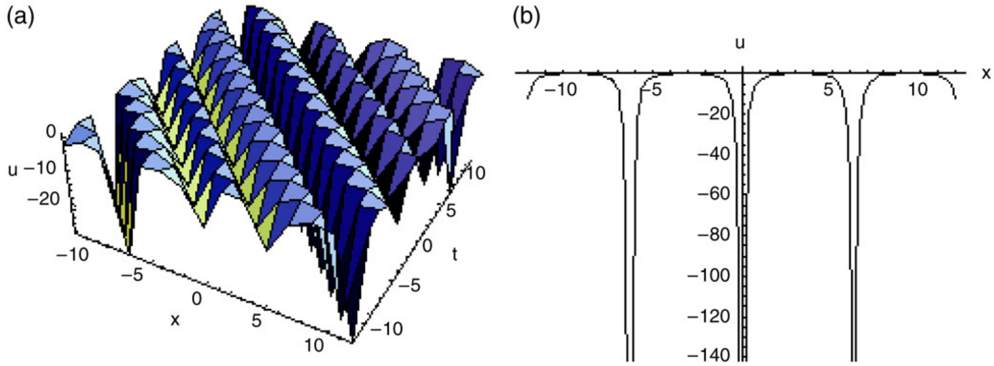

Fig. 6. (a) Periodic wave solutions of u_6 when $a = 1, b = 0, c = 1, a_0 = -\frac{1}{6}, \xi_0 = 0, k = 0.5, q = 1$. (b) Plane graph when $t = 0$.

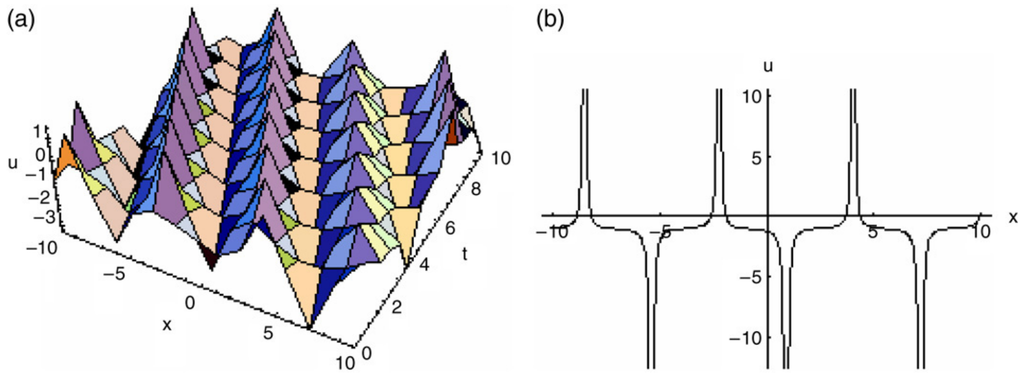

Fig. 7. (a) Periodic wave solutions of u_6 when $a = 5, b = 4, c = 3, a_0 = -\frac{1}{6}, \xi_0 = 0, k = 0.5, q = 1$. (b) Plane graph when $t = 0$.

We can give the numerical simulation of u_5, u_6, and v_7 (see Figs. 5–8).

Remark 2. It is easy to obtain bell-shaped solitary wave solutions, singular wave solutions and many other types of periodic wave solutions if we choose the special value of $a, b, c, a_0, b_0, c_0, k, r, \varepsilon, q$ from $u_i, v_i, w_i, (i = 1, \ldots, 7)$ for the system (3.2).

To our knowledge, the seven types of explicit solutions we obtained here are new.

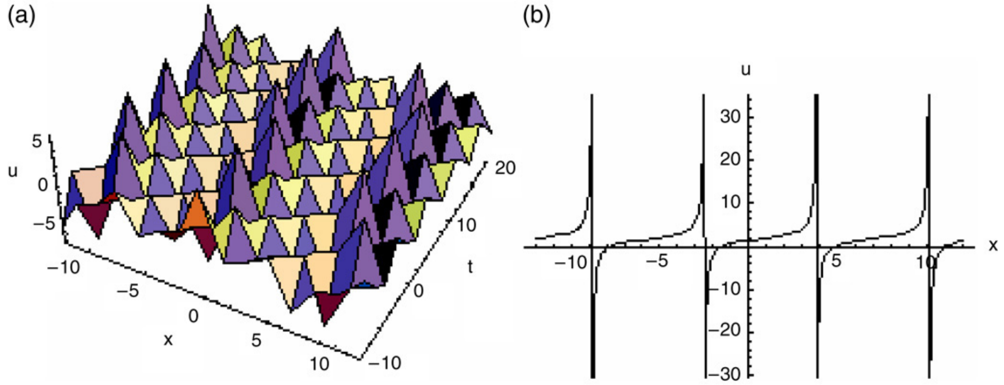

Fig. 8. (a) Periodic wave solutions of v_7 when $a = 5$, $b = 4$, $c = 3$, $a_0 = 0.5$, $b_0 = 0$, $\xi_0 = 0$, $k = 0.5$, $q = 1$. (b) Plane graph when $t = 0$.

4. Conclusion

In this paper, based on two new Riccati equations, we generalized the Riccati method and applied it to find many exact solitary wave solutions and periodic solutions for the system (3.2) and it is easy to see that the new method is more general and simplified than the extended method in [8] and the variable-coefficient projective Riccati equation method in [3]; [3–11] are the case for our method. This paper has shown that the new projective Riccati equation method is sufficient incentive to seek more new exact soliton solutions of NEEs in mathematical physics.

Acknowledgements

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